

Quantum channel detection

C. Macchiavello and M. Rossi

Dipartimento di Fisica and INFN-Sezione di Pavia, via Bassi 6, 27100 Pavia, Italy

(Dated: August 28, 2012)

We present a method to detect properties of quantum noisy channels, assuming that some a priori information about the form of the channel is available. The method is based on a correspondence with entanglement detection methods for multipartite density matrices based on witness operators. We illustrate the method in the case of entanglement breaking channels and non separable random unitary channels, and show how it can be implemented experimentally by means of local measurements.

The possibility of determining properties of quantum communication channels or quantum devices is of great importance in order to be able to design and operate the channel at the best of its performances. In many realistic implementations some a priori information on the form of a quantum channel, or a quantum noise process, is available and it is of large interest to determine experimentally whether or not the channel has a certain property. The aim of this work is to propose efficient methods to detect this possibility by avoiding full quantum process tomography, which allows a complete reconstruction of the channel but it requires a large number of measurement settings. At the same time, from the point of view of implementations, our procedure is experimentally feasible with present day technology based on local measurements.

Quantum channels, and in general quantum noise processes, are described by completely positive and trace preserving (CPT) maps \mathcal{M} , which can be expressed in the Kraus form [1] as

$$\mathcal{M}[\rho] = \sum_k A_k \rho A_k^\dagger, \quad (1)$$

where ρ is the density operator of the quantum system on which the channel acts and the Kraus operators $\{A_k\}$ fulfil the constraint $\sum_k A_k^\dagger A_k = \mathbf{1}$.

In order to develop the detection method proposed, we will use the Choi-Jamolkowski isomorphism, which gives a one-to-one correspondence between CPT maps acting on $\mathcal{D}(\mathcal{H})$ (the set of density operators on \mathcal{H}) and bipartite density operators $C_{\mathcal{M}}$ on $\mathcal{H} \otimes \mathcal{H}$. This isomorphism can be described as

$$\mathcal{M} \iff C_{\mathcal{M}} = \mathcal{M} \otimes \mathcal{I}[|\alpha\rangle\langle\alpha|], \quad (2)$$

where \mathcal{I} is the identity map, and $|\alpha\rangle$ is the maximally entangled states with respect to the bipartite space $\mathcal{H} \otimes \mathcal{H}$, i.e. $|\alpha\rangle = \frac{1}{\sqrt{d}} \sum_{k=1}^d |k\rangle|k\rangle$ (we consider here quantum channels acting on systems with finite dimension d). This is schematically depicted in Fig. 1.

In the Kraus operators description the above relation

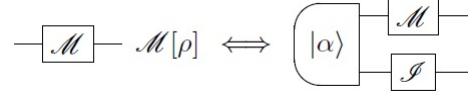


Figure 1. Scheme showing how the Choi-Jamolkowski isomorphism works, on the left the map \mathcal{M} , on the right the corresponding Choi state $C_{\mathcal{M}}$.

takes the form

$$C_{\mathcal{M}} = \sum_k (A_k \otimes \mathbf{1}) |\alpha\rangle\langle\alpha| (A_k^\dagger \otimes \mathbf{1}). \quad (3)$$

In this work, by the above isomorphism, we link some specific properties of quantum channels to properties of the corresponding Choi states $C_{\mathcal{M}}$. Therefore we start from state detection methods to construct methods for quantum channels detection. In particular, we find a connection between quantum channel properties and (multipartite) entanglement properties of the corresponding Choi states. We will consider properties that are based on a convex structure of the quantum channels.

Consider as a first case the class of entanglement breaking channels [2]. A possible definition for an entanglement breaking channel is based on the separability of its Choi state. More explicitly, a quantum channel is entanglement breaking if and only if its Choi state is separable. The set of Choi states corresponding to entanglement breaking channels therefore contains only bipartite separable states. This allows to formulate a method to detect whether a quantum channel is not entanglement breaking by exploiting entanglement detection methods designed for bipartite systems [3]. To this end, we remind the concept of entanglement detection via witness operators [4]: a state ρ is entangled if and only if there exists a hermitian operator W such that $\text{Tr}[W\rho] < 0$ and $\text{Tr}[W\rho_{sep}] \geq 0$ for all separable states. As a simple example of quantum channel detection consider the case of two-dimensional systems and the single qubit depolaris-

ing channel, defined as

$$\Gamma_{\{p\}}[\rho] = \sum_{i=0}^3 p_i \sigma_i \rho \sigma_i \quad (4)$$

where σ_0 is the identity operator, $\{\sigma_i\}$ ($i = 1, 2, 3$) are the three Pauli operators $\sigma_x, \sigma_y, \sigma_z$ respectively (for brevity of notation in the following the Pauli operators will be denoted by X, Y and Z), and $p_0 = 1 - p$ (with $p \in [0, 1]$), while $p_i = p/3$ for $i = 1, 2, 3$. Such a channel is entanglement breaking for $p \geq 1/2$. The corresponding set of Choi bipartite density operators is given by the Werner states

$$\rho_p = (1 - \frac{4}{3}p)|\alpha\rangle\langle\alpha| + \frac{p}{3}\mathbf{1}. \quad (5)$$

It is then possible to detect whether a depolarising channel is not entanglement breaking by exploiting an entanglement witness operator for the above set of states [3, 5], which has the form

$$W_{EB} = \frac{1}{4}(\mathbf{1} \otimes \mathbf{1} + X \otimes X - Y \otimes Y - Z \otimes Z). \quad (6)$$

The method can then be implemented by preparing a two-qubit state in the maximally entangled state $|\alpha\rangle$, then operating with the quantum channel to be detected on one of the two qubits and measuring the operator W_{EB} at the end. As we can see from the above form, the operator (6) can be measured by performing a set of three local measurements performed on the two qubits. If the resulting average value is negative, we can then conclude that the channel under consideration is not entanglement breaking.

We will now consider a different scenario, namely the case of random unitary channels (RU). These are defined as

$$\mathcal{U}[\rho] = \sum_k p_k U_k \rho U_k^\dagger, \quad (7)$$

where U_k are unitary operators and $p_k \geq 0$ with $\sum_k p_k = 1$. Notice that this kind of maps includes several interesting models of quantum noisy channels, such as the already mentioned depolarising channel or the phase damping channel and the bit flip channel [6]. Random unitaries were also studied extensively and characterised in Ref. [7].

Let us now assume that the system on which the random unitary channel acts is a bipartite system ρ_{AB} (composed of systems A and B). We can then identify a class of random unitary maps which is separable, namely that can be written in the form

$$\mathcal{V}[\rho_{AB}] = \sum_k p_k (V_{k,A} \otimes W_{k,B}) \rho (V_{k,A}^\dagger \otimes W_{k,B}^\dagger), \quad (8)$$

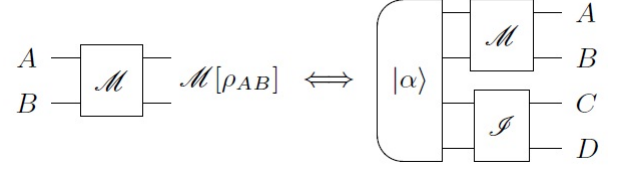


Figure 2. Scheme showing how the Choi-Jamolkowski isomorphism works in the case of four-partite states. The state $|\alpha\rangle$ on the right is the maximally entangled state with respect to the bipartition AB—CD.

where ρ_{AB} is a bipartite system, and both $V_{k,A}$ and $W_{k,B}$ are unitary operators for all k 's, acting on systems A and B respectively. Quantum channels of the above form are named separable random unitaries (SRU) and they form a convex subset in the set of all CPT maps acting on bipartite systems. Interesting examples of channels of this form are given by Pauli memory channels [8].

When considering quantum channels acting on bipartite systems, the Choi state is a four-partite state (composed of systems A, B, C and D), as shown in Fig. 2. Notice that the state $|\alpha\rangle = \frac{1}{\sqrt{d_{AB}}} \sum_{k,j=1}^{d_{AB}} |k,j\rangle_{AB} |k,j\rangle_{CD}$ (where $d_{AB} = d_A d_B$ is now the dimension of the Hilbert space of the bipartite system AB), can also be written as $|\alpha\rangle = |\alpha\rangle_{AC} |\alpha\rangle_{BD}$, namely it is a biseparable state for the partition AC—BD of the global four-partite system. The Choi states corresponding to SRU channels therefore form a convex set, which is a subset of all biseparable states for the partition AC—BD. Since the extremal maps of the set of separable random unitaries are given by local unitaries $U_A \otimes U_B$, the extremal bipartite pure states in the corresponding set of Choi states have the form (we name this set of four-partite density operators S_{SRU})

$$(U_A \otimes \mathbf{1}_C) |\alpha\rangle_{AC} \otimes (U_B \otimes \mathbf{1}_D) |\alpha\rangle_{BD}. \quad (9)$$

We can now detect non separable RU maps (which correspond to Choi states that are entangled in the bipartition AC—BD) by designing suitable witness operators that detect the corresponding Choi state with respect to biseparable states belonging to S_{SRU} in the bipartition AC—BD.

We illustrate this procedure with a simple example. Consider the case of detecting a CNOT operation, which is a non separable unitary operation acting on two qubits. The corresponding Choi state has the form

$$\begin{aligned} C_{\text{CNOT}} &= |\text{CNOT}\rangle\langle\text{CNOT}| \\ &= (\text{CNOT} \otimes \mathbf{1}) |\alpha\rangle\langle\alpha| (\text{CNOT} \otimes \mathbf{1}), \end{aligned} \quad (10)$$

where the CNOT operation is given by

$$\text{CNOT} = \begin{pmatrix} 1 & 0 \\ 0 & X \end{pmatrix}, \quad (11)$$

with $\mathbf{1}$ representing the 2×2 identity matrix, and X the usual Pauli operator. A suitable detection operator for the CNOT gate can be constructed as

$$W_{\text{CNOT}} = \beta \mathbf{1} - C_{\text{CNOT}}, \quad (12)$$

where the coefficient β is the squared overlap between the closest biseparable state in the set S_{SRU} and the entangled state C_{CNOT} , namely

$$\beta = \max_{|\phi\rangle \in S_{SRU}} \langle \phi | C_{\text{CNOT}} | \phi \rangle. \quad (13)$$

Notice that, since the maximum of a linear function over a convex set is always achieved on the extremal points, the maximum above can be always calculated by maximising over the pure biseparable states (9). Furthermore, the coefficient β used above is the smallest value of the coefficient leading to a semi-positive expectation value of W_{CNOT} over all biseparable states, so W_{CNOT} is clearly optimal among the witnesses of this form.

The task is now to compute the coefficient β . This task can be accomplished as follows, by exploiting techniques to calculate the amount of entanglement in bipartite systems. The state (10) is not separable with respect to the split AC—BD and it can be expressed in the Schmidt decomposition regarding that split as

$$|\text{CNOT}\rangle = \frac{1}{\sqrt{2}}(|00\rangle_{AC}|\alpha\rangle_{CD} + |11\rangle_{AC}|\psi^+\rangle_{CD}), \quad (14)$$

where $|\psi^+\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$. The above expression naturally proves that the maximum overlap with any biseparable state w.r.t. AC—BD cannot exceed the value of $1/2$. Since the convex set S_{SRU} of allowed states in our optimisation problem is smaller than the set of all biseparable states, this would give us only an upper bound for the maximum overlap β . However, two local unitary operations U_A and U_B that saturate this bound can be explicitly found, namely

$$U_A = S, \quad (15)$$

$$U_B = e^{-i\frac{\pi}{4}X}, \quad (16)$$

where S is the phase gate given by $S = \text{diag}(1, i)$. This finally proves that the optimal coefficient β equals $1/2$ also if we restrict to the set of biseparable states S_{SRU} . Therefore, the detection operator W_{CNOT} can be decomposed into a linear combination of local operators as follows

$$\begin{aligned} W_{\text{CNOT}} = \frac{1}{64} & (3\mathbf{1}\mathbf{1}\mathbf{1}\mathbf{1}\mathbf{1} - \mathbf{1}X\mathbf{1}X - XXX\mathbf{1} - X\mathbf{1}XX \\ & - ZZ\mathbf{1}Z + ZY\mathbf{1}Y + YYXZ + YZXY \\ & - Z\mathbf{1}Z\mathbf{1} - ZXZX + YXY\mathbf{1} + Y\mathbf{1}YX \\ & - \mathbf{1}ZZZ + \mathbf{1}YZY + XYYZ + XZYY), \end{aligned} \quad (17)$$

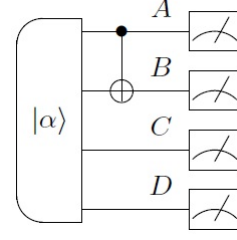


Figure 3. Experimental scheme implementing the detection of the CNOT gate.

where for simplicity of notation the tensor product symbol has been omitted. As we can see from the above form, the CNOT can be detected by using nine different local measurements settings, namely $\{XXXX, ZZZZ, ZYZY, YXYX, YYXZ, YZXY, ZXZX, XYYZ, XZYY\}$. Following [3, 9], it can be also easily proved that the above form is optimal in the sense that it involves the smallest number of measurement settings. From an experimental point of view, the optimal detection procedure then works as follows: prepare a four-partite qubit system in state $|\alpha\rangle = |\alpha\rangle_{AC}|\alpha\rangle_{BD}$, input qubits A and B to the quantum channel and finally perform the set of nine local measurements reported above in order to measure the operator (17). If the resulting average value is negative then the quantum channel is detected as a non separable random unitary map. The experimental scheme is shown in Fig. 3.

Notice that the number of measurements needed in this scheme is much smaller than the one required for complete quantum process tomography [6], that scales as d_{AB}^4 . The number of measurement settings in the detection scheme can be further decreased if we allow a non optimal detection operator, in the sense that the coefficient β in (12) is smaller than the maximum value. In this case, since the state C_{CNOT} is a stabilizer state with generators $\{XXX\mathbf{1}, \mathbf{1}X\mathbf{1}X, Z\mathbf{1}Z\mathbf{1}, ZZ\mathbf{1}Z\}$, an alternative detection operator can be derived, following the approach of Ref. [10]. The resulting suboptimal detection operator turns out to be

$$\begin{aligned} \tilde{W}_{\text{CNOT}} = 3\mathbf{1} - 2 & \left[\frac{(\mathbf{1} + XXX\mathbf{1})(\mathbf{1} + \mathbf{1}X\mathbf{1}X)}{2} \right. \\ & \left. + \frac{(\mathbf{1} + Z\mathbf{1}Z\mathbf{1})(\mathbf{1} + ZZ\mathbf{1}Z)}{2} \right], \end{aligned} \quad (18)$$

which requires only the two local measurement settings $\{XXXX, ZZZZ\}$.

We will now study the robustness of the method in the presence of additional noise, which can influence the operation of the quantum channel. Let us first assume that phase damping noise is present, acting independently on the two qubits A and B in general both before and after the operation of the CNOT gate, as shown in Fig.

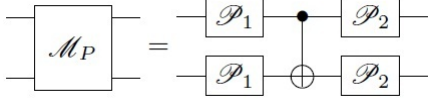


Figure 4. CNOT gate in the presence of dephasing noise.

4. Phase damping noise is described by a CPT map of the form (4), with probabilities $p_0 = q$, $p_1 = p_2 = 0$ and $p_3 = 1 - q$. Notice that the global resulting channel shown in Fig. 4 is still a random unitary channel.

In order to quantify the noise robustness of the witness W_{CNOT} with respect to phase damping noise, we calculate the expectation value of W_{CNOT} given by (12) (with $\beta = 1/2$) with respect to the bipartite state $C_{\mathcal{M}_P}$ depicted in Fig. 4, i.e. the Choi state corresponding to the composite map $\mathcal{M}_P = (\mathcal{P}_2 \otimes \mathcal{P}_2)\text{CNOT}(\mathcal{P}_1 \otimes \mathcal{P}_1)$. The problem thus reduces to evaluate the overlap between the Choi states C_{CNOT} and $C_{\mathcal{M}_P}$. This can be easily achieved as follows. In general, given two maps \mathcal{M} and \mathcal{L} acting on a d -dimensional system and described by sets of Kraus operators $\{A_k\}$ and $\{B_l\}$ respectively, the overlap between the two corresponding Choi states takes the simple form

$$\text{Tr}[C_{\mathcal{M}}C_{\mathcal{L}}] = \frac{1}{d^2} \sum_{k,l} |\text{Tr}[A_k^\dagger B_l]|^2, \quad (19)$$

where the double summation is over the Kraus operators. In our case, the two maps correspond to the CNOT gate and the composite map \mathcal{M}_P defined above. After a lengthy calculation, this procedure leads to

$$\text{Tr}[W_{\text{CNOT}}C_{\mathcal{M}_P}] = \frac{1}{2} - [q_1^2 q_2^2 + (1 - q_1)(1 - q_2)(q_1 + q_2 - q_1 q_2)]. \quad (20)$$

From the above expression we can see that $\text{Tr}[W_{\text{CNOT}}C_{\mathcal{M}_P}] < 0$ for certain intervals of the noise parameters q_1 and q_2 . When the dephasing channels introduce the same level of noise ($q_1 = q_2 = q$) the expectation value of W_{CNOT} turns out to be negative for $q > 0.83$ and therefore the CNOT operation can be detected in this interval. As we can see, the presence of local dephasing noise affects the CNOT operation in such a way that, above a certain amount of noise, the noisy CNOT operation becomes separable.

We have also evaluated noise robustness for the amplitude damping channel, which is not a random unitary noise and it is described by the following Kraus operators

$$A_1 = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-\gamma} \end{pmatrix}, A_2 = \begin{pmatrix} 0 & \sqrt{\gamma} \\ 0 & 0 \end{pmatrix}, \quad (21)$$

where γ is the parameter characterising the damping. In this case, following the same procedure described above

by considering now the composite map $\mathcal{M}_A = (\mathcal{A}_2 \otimes \mathcal{A}_2)\text{CNOT}(\mathcal{A}_1 \otimes \mathcal{A}_1)$, we have

$$\text{Tr}[W_{\text{CNOT}}C_{\mathcal{M}_A}] = \frac{1}{2} - \frac{1}{16} [(1 + \sqrt{\gamma_1 \gamma_2}(1 + \sqrt{\gamma_1} + \sqrt{\gamma_2}))^2 + \gamma_1 \bar{\gamma}_1 \gamma_2 \bar{\gamma}_2], \quad (22)$$

where we have defined $\bar{\gamma} = 1 - \gamma$. For the particular case of $\gamma_1 = \gamma_2 = \gamma$ we have that the above expression is negative for $\gamma < 0.31$, and therefore the composite map can be detected as a non separable random unitary in this range of noise parameter γ .

In conclusion, we have presented an experimentally feasible method to detect specific properties of noisy quantum channels, including quantum noise processes. The proposed procedure works when some a priori knowledge on the quantum channel is available and is based on a link to detection methods for entanglement properties of multipartite quantum states via witness operators. In this work the method has been explicitly illustrated to detect entanglement breaking or separable random unitary properties of quantum channels. The advantage over standard quantum process tomography is that a much smaller number of measurement settings is needed in an experimental implementation. Moreover, we want to point out that the proposed scheme relies on local measurements and it is achievable with current technology, for example in quantum optical implementations [11].

-
- [1] K. Kraus, *States, effects and operations*, (Springer, Berlin, 1983).
 - [2] M. Horodecki, P.W. Shor and M.B. Ruskai, Rev. Math. Phys. **15**, 629 (2003).
 - [3] O. Ghne, P. Hyllus, D. Bru, A. Ekert, M. Lewenstein, C. Macchiavello and A. Sanpera, Phys. Rev. A **66**, 062305 (2002).
 - [4] M. Horodecki, P. Horodecki and R. Horodecki, Phys. Lett. A **223**, 1 (1996); B.M. Terhal, Phys. Lett. A **271**, 319 (2000).
 - [5] O. Ghne, P. Hyllus, D. Bru, A. Ekert, M. Lewenstein, C. Macchiavello and A. Sanpera, J. Mod. Opt. **50**, 1079 (2003).
 - [6] See, for example M.A. Nielsen and I.L. Chuang, *Quantum computation and quantum information*, Cambridge University Press, Cambridge (2000).
 - [7] K.M.R. Audenaert and S. Scheel, New J. Phys. **10**, 023011 (2008).
 - [8] C. Macchiavello and G.M. Palma, Phys. Rev. A **65**, 050301(R) (2002).
 - [9] O. Ghne, P. Hyllus, Int. J. Theor. Phys. **42**, 1001 (2003).
 - [10] G. Toth and O. Ghne, Phys. Rev. Lett. **94**, 060501 (2005).
 - [11] See for example A. Chiuri *et al.*, Phys. Rev. Lett. **105**, 250501 (2010); A. Chiuri *et al.*, Phys. Rev. Lett. **107**, 253602 (2011).